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Characters, dimensions and branching rules for covariant irreps of U(*N*/*M*)

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Abstract. The theory of symmetric functions is used to obtain simple expressions for the characters and dimensions of the covariant irreps of the supergroup $U(N/M)$. Schur function expressions are given for a number of important branching rules.

1. Introduction

Balantekin and Bars (1981a, b, 1982) have recently outlined methods for evaluating the characters, dimensions and branching rules for the covariant irreps of $SU(N/M)$ supergroups. Their results may be succinctly expressed by making use of the theory of symmetric functions. The relevant theory of symmetric functions has been given by Macdonald (1979).

In this note we give simple expressions for the characters and dimensions of the covariant irreps of $SU(N/M)$ together with simple Schur function expressions for appropriate branching rules. We follow the notation given in the papers of Balantekin and Bars together with that of the relevant theory of Schur functions and series (Littlewood 1950, Wybourne 1970, Black and Wybourne 1983, Black *et al* 1983, King *et al* 1981).

2. Characters and dimensions

The covariant irreps for both $SU_{(N)}$ and $SU_{(N/M)}$ may be conveniently labelled by ordered partitions $\{\lambda\}$. For $SU_{(N)}$ restriction to inequivalent irrep limits the partitions to not more than $N-1$ parts; no such restriction arises for $SU(N/M)$.

Schur functions $\{\lambda\}$ arise naturally in the theory of the symmetric group S_n with the characteristics χ^λ_ρ of S_n being the elements of the transition matrix $M(s,p)_{\lambda\rho}$ that relates the Schur functions $\{\lambda\}$ to the power sum symmetric functions p_ρ to give (Macdonald 1979)

$$\{\lambda\}=\sum_\rho M(s,p)_{\lambda\rho}p_\rho=\frac{1}{\omega_\lambda!}\sum_\rho h_\rho\chi^\lambda_\rho p_\rho \tag{1}$$

where ω_λ is the sum of the parts of the partition (λ) and

$$(\rho)=(k^{m_k}\ \dots\ i^{m_i}\ \dots\ 2^{m_2}\ 1^{m_1}) \tag{2}$$

in the class ρ of h_ρ elements of S_n $(n=\omega_\lambda=\omega_\rho)$ with

$$h_\rho = n!\, z_\rho^{-1} \tag{3}$$

where

$$z_\rho = \prod_{i=1}^{k} i^{m_i} m_i!. \tag{4}$$

The class ρ is of length l_ρ and parity ε_ρ where

$$l_l = \sum_{i=1}^{k} m_i \qquad \text{and} \qquad \varepsilon_\rho = (-1)^{\omega_\rho - l_\rho}. \tag{5}$$

It is convenient to let m_ρ^{o} (m_ρ^{e}) denote the sum of the multiplicities m_i of the odd (even) parts of ρ.

If the symmetric functions p_ρ are defined on the roots of the group elements U of $U_{(N)}$ we have for the character $\chi_{\{\lambda\}}$ of U_N

$$\chi_{\{\lambda\}} = \frac{1}{\omega_\lambda!} \sum_\rho h_\rho \chi_\rho^\lambda \prod_{i=1}^{k} (\operatorname{Tr} U^i)^{m_i}. \tag{6}$$

For $U_{(N/M)}$ we simply replace the trace Tr by the supertrace STr (Balantekin and Bars 1981a) to give for $U_{(N/M)}$

$$\chi_{\{\lambda\}} = \frac{1}{\omega_\lambda!} \sum_\rho h_\rho \chi_\rho^\lambda \prod_{i=1}^{k} (\operatorname{STr} U^i)^{m_i}. \tag{7}$$

The dimension $D_{\{\lambda\}}$ of $\{\lambda\}$ for $U_{(N)}$ then becomes

$$D_{\{\lambda\}} = \frac{1}{\omega_\lambda!} \sum_\rho h_\rho \chi_\rho^\lambda N^{l_\rho} \tag{8}$$

while for the class I irreps $\{\lambda\}$ of $U_{(N/M)}$

$$D_{\{\lambda\}} = \frac{1}{\omega_\lambda!} \sum_\rho h_\rho \chi_\rho^\lambda x^{m_\rho^{\text{o}}} y^{m_\rho^{\text{e}}} \tag{9}$$

and for the class II irreps $\overline{\{\lambda\}}$

$$D_{\overline{\{\lambda\}}} = \frac{1}{\omega_\lambda!} \sum_\rho h_\rho \chi_\rho^\lambda x^{m_\rho^{\text{o}}} (-y)^{m_\rho^{\text{e}}} \tag{10}$$

where we have put

$$x = N + M \qquad \text{and} \qquad y = N - M. \tag{11}$$

Thus the expressions for $D_{\{\lambda\}}$ for $U_{(N/M)}$ become simple polynomials in x and y that may be directly read off the S_n character tables (Littlewood 1950) to give the results shown in table 1. It follows from the comparison of (9) and (10) that the dimensions $D_{\overline{\{\lambda\}}}$ may be obtained from those of $D_{\{\lambda\}}$ by simply replacing y by $-y$ in table 1.

Since for S_n

$$\chi_\rho^{\tilde{\lambda}} = \chi_\rho^{1^n} \chi_\rho^\lambda = \varepsilon_\rho \chi_\rho^\lambda \tag{12}$$

(with $\omega_\lambda = n$) we can obtain the dimensions $D_{\{\tilde{\lambda}\}}$ from the polynomial expressions for $D_{\{\lambda\}}$ by making the replacements

$$x^k y^l \to (-1)^{n+k+l} x^k y^l. \tag{13}$$

Table 1. Dimensions of covariant irreps $\{\lambda\}$ of $U_{(N/M)}$.

$\{\lambda\}$	$D_{\{\lambda\}}$
$\{1\}$	x
$\{2\}$	$(1/2!)(x^2+y)$
$\{3\}$	$(x/3!)(x^2+3y+2)$
$\{21\}$	$(2x/3!)(x^2-1)$
$\{4\}$	$(1/4!)(x^4+6x^2y+8x^2+3y^2+6y)$
$\{31\}$	$(3/4!)(x^4+2x^2y-y^2-2y)$
$\{2^2\}$	$(2/4!)(x^4-4x^2+3y^2)$
$\{5\}$	$(x/5!)(x^4+10x^2y+20xy+15y^2+50y+24)$
$\{41\}$	$(4x/5!)(x^4+5x^2y+5x^2-5y-6)$
$\{32\}$	$(5x/5!)(x^4+2x^2y-4x^2+3y^2-2y)$
$\{31^2\}$	$(6x/5!)(x^4-5y^2+4)$
$\{6\}$	$(1/6!)(x^6+15x^4y+40x^4+45x^2y^2+15y^3+210x^2y+184x^2+90y^2+120y)$
$\{51\}$	$(5/6!)(x^6+9x^4y+16x^4+9x^2y^2-3y^2+18x^2y-8x^2-18y^2-24y)$
$\{42\}$	$(9/6!)(x^6+5x^4y+5x^2y^2-10x^2y+5y^3-10y^2-16x^2)$
$\{41^2\}$	$(10/6!)(x^6+3x^4y+4x^4-9x^2y^2-12x^2y-3y^3+4x^2+12y)$
$\{3^2\}$	$(5/6!)(x^6+3x^4y-8x^4+9x^2y^2+6x^2y-9y^3+16x^2-18y^2)$
$\{321\}$	$(16/6!)(x^4-5x^2+4)$
$\{7\}$	$(x/7!)(x^6+21x^4y+70x^4+105x^2y^2+630x^2y^2+105y^3+784x^2+840y^2+1764y+720)$
$\{61\}$	$(6x/7!)(x^6+14x^4y+35x^4+35x^2y^2+140x^2y+84x^2-35y^2-154y-120)$
$\{52\}$	$(14x/7!)(x^6+9x^4y+10x^4+15x^2y^2-56x^2+15y^3+15y^2-24y)$
$\{51^2\}$	$(15x/7!)(x^6+7x^4y+14x^4-7x^2y^2-14x^3y-21y^3-56y^2+28y+48)$
$\{43\}$	$(14x/7!)(x^6+6x^4y-5x^4+15x^2y^2+4x^2-15y^2-6y)$
$\{421\}$	$(35x/7!)(x^6+3x^4y-2x^4-3x^2y^2-18x^2y+3y^3+12y^2-8x^2+12y)$
$\{41^3\}$	$(20x/7!)(x^6+7x^4-21x^2y^2+14x^2+21y^2-36)$
$\{3^21\}$	$(21x/7!)(x^6+x^4y-10x^4+5x^2y^2+10x^2y-15y^3+24x^2-20y^2+4y)$
$\{8\}$	$(1/8!)(x^8+28x^6y+112x^6+210x^4y^2+1540x^4y+2464x^4+420x^2y^3+4200x^2y^2+11872x^2y$ $+105y^4+1260y^3+4620y^2+8448x^2+5040y)$
$\{71\}$	$(7/8!)(x^8+20x^6y+64x^6+90x^4y^2+500x^4y+544x^4+60x^2y^3+360x^2y^2+320x^2y-384x^2$ $-15y^4-180y^3-660y^2-720y)$
$\{62\}$	$(20/8!)(x^8+14x^6y+28x^6+42x^4y^2+98x^4y-56x^4+42x^2y^3+84x^2y^2-280x^2y-288x^2+21y^4$ $+126y^3+168y^2)$
$\{61^2\}$	$(21/8!)(x^8+12x^6y+32x^6+10x^4y^2+40x^4y+64x^4-60x^2y^3-280x^2y^2-192x^2y+128x^2-15y^4$ $-60y^3+60y^2+240y)$
$\{53\}$	$(28/8!)(x^8+10x^6y+4x^6+30x^4y^2+10x^4y-56x^4+30x^2y^2+60x^2y^2+40x^2y+96x^2-15y^4$ $-90y^3-120y^2)$
$\{521\}$	$(640x^2/8!)(x^6+7x^4y+7x^4-35x^2y-56x^2+28y+48)$
$\{51^3\}$	$(35/8!)(x^8+4x^6y+16x^6-30x^4y^2-20x^4y+64x^4-24x^2y^2-36x^2y^3+160x^2y+9y^4-36y^3$ $-36y^2-144y)$
$\{4^2\}$	$(14/8!)(x^8+8x^6y-8x^6+30x^4y^2+20x^4y+64x^4-120x^2y^2-208x^2y-384x^2+45y^4+180y^3$ $+180y^2)$
$\{431\}$	$(70/8!)(x^8+4x^6y-8x^6+6x^4y^3-8x^4y-12x^2y^3-24x^2y^2+16x^2y+16x^4-3y^4+12y^2)$
$\{42^2\}$	$(56/8!)(x^8+2x^6y-8x^6-40x^4y+30x^2y^3+8x^2y+4x^4+48x^2+5y^4-60y^2)$
$\{421^2\}$	$(90/8!)(x^8-14x^4y^2+56x^2y^2-64x^2-7y^4+28y^2)$
$\{3^22\}$	$(42/8!)(x^8-16x^6+10x^4y^2+64x^4-40x^2y^2-64x^2-15y^4+60y^2)$

The superdimension $S_{\{\lambda\}}$ of the irrep $\{\lambda\}$ of $U_{(N/M)}$ is defined as the difference $B-F$ between the boson number B and the fermion number F and may be obtained from $D_{\{\lambda\}}$ by replacing x and y in table 1 or by replacing N by y in (8), showing that $S_{\{\lambda\}}$ is simply the dimension $D_{\{\lambda\}}$ for $U_{(N-M)}$. Similarly we have

$$S_{\overline{\{\lambda\}}} = -S_{\{\lambda\}} \tag{14}$$

corresponding to interchanging the number of bosonic and fermionic components in going from class I to class II irreps. The superdimension $S_{\{\tilde{\lambda}\}}$ is simply the dimension $D_{\{\tilde{\lambda}\}}$ for $U_{(N-M)}$. Thus, to find $S_{\{\tilde{\lambda}\}}$ from table 1, make the replacement given by (13) and then replace x by y.

By way of example we have from table 1 for the $\{41\}$ irrep of $U_{(N/M)}$

$$D_{\{41\}} = (4x/5!)(x^4+5x^2y+5x^2-5y-6)$$

and from (13)

$$D_{\{\widetilde{41}\}} = D_{\{21^3\}} = (4x/5!)(x^4-5x^2y+5x^2+5y-6)$$

and coincidently $D_{\{\widetilde{41}\}} = D_{\overline{\{41\}}}$. Similarly we have

$$S_{\{41\}} = (4y/5!)(y^4+5y^3+5y^2-5y-6)$$

with

$$S_{\overline{\{41\}}} = -S_{\{41\}}$$

and

$$S_{\{\widetilde{41}\}} = (4y/5!)(y^4-5y^3+5y^2+5y-6).$$

These results hold for all N and M. Specialising to particular values of N and M, we have for $U_{(2/1)}$

$$D_{\{41\}} = 16, \qquad D_{\{\widetilde{41}\}} = D_{\{41\}} = D_{\{21^3\}} = 8,$$

$$S_{\{41\}} = S_{\overline{\{41\}}} = S_{\{\widetilde{41}\}} = S_{\{21^3\}} = 0,$$

whereas for $U_{(5/2)}$ we have

$$D_{\{41\}} = 784, \qquad D_{\{\widetilde{41}\}} = D_{\{21^3\}} = D_{\overline{\{41\}}} = 448,$$

$$S_{\{41\}} = 24, \qquad S_{\overline{\{41\}}} = -24, \qquad S_{\{21^3\}} = 0.$$

3. Branching rules

Balantekin and Bars have given procedures for obtaining branching rules. Some rules using Schur function methods have also been published (Delbourgo and Jarvis 1983, Dondi and Jarvis 1980, 1981, King 1982). The results given by Balantekin and Bars may be succinctly expressed in terms of Schur function operations. A number of important branching rules are summarised in table 2 with the Schur function series notation following that of Black *et al* (1983). The modification rules for the relevant subgroups have been given by Black *et al* (1983). Noting the branching rules for $SU_{(N/M)} \downarrow OSp_{(N/M)}$ and $SU_{(N/M)} \downarrow SpO_{(N/M)}$, we can write the dimensions of the irrep of $OSp_{(N/M)}$ and $SpO_{(N/M)}$ in terms of those of $SU_{(N/M)}$ to give

$$D_{[\lambda]} = D_{\{\lambda/C\}} \qquad \text{and} \qquad D_{\langle\lambda\rangle} = D_{\{\lambda/A\}} \tag{15}$$

respectively. The dimensions for $\omega_\lambda \leq 4$ of $OSp_{(N/M)}$ and $SpO_{(N/M)}$ are listed in tables 3 and 4 respectively as polynomials in x and y. In each case the superdimension may be found by letting $x \to y$, giving the dimensions of $[\lambda]$ or $\langle\lambda\rangle$ for $O_{(N-M)}$ and $Sp_{(N-M)}$ respectively.

Table 2. Branching rules for covariant irreps.

$U_{(N/M)} \downarrow U_N \times U_M$	
$\{\lambda\}$	$\downarrow \sum_\zeta \{\lambda/\zeta\} \times \{\tilde{\zeta}\}$
$U_{(N_1+N_2/M_1+M_2)} \downarrow U_{(N_1/M_1)} \times U_{(N_2/M_2)}$	
$\{\lambda\}$	$\downarrow \sum_\zeta \{\lambda/\zeta\} \times \{\zeta\}$
$U_{(N_1N_2+M_1M_2/N_1M_2+N_2M_1)} \downarrow U_{(N_1/M_1)} \times U_{(N_2/M_2)}$	
$\{\lambda\}$	$\downarrow \sum_\zeta \{\lambda \circ \zeta\} \times \{\zeta\}$
$SU_{(N/M)} \downarrow SU_{(N)} \times SU_{(M)} \times U_{(1)}$	
$\{\lambda\}$	$\downarrow \sum_\zeta \{\lambda/\zeta\} \times \{\tilde{\zeta}\} \times \left\{\frac{\omega_\lambda - \omega_\zeta}{N} + \frac{\omega_\zeta}{M}\right\}$
$SU_{(N_1+N_2/M_1+M_2)} \downarrow SU_{(N_1/M_1)} \times SU_{(N_2/M_2)} \times U_{(1)}$	
$\{\lambda\}$	$\downarrow \sum_\zeta \{\lambda/\zeta\} \times \{\zeta\} \times \left\{\frac{\omega_\lambda - \omega_\zeta}{N_1 - M_1} - \frac{\omega_\zeta}{N_2 - M_2}\right\}$
$SU_{(N/M)} \downarrow OSp_{(N/M)}$	
$\{\lambda\}$	$\downarrow [\lambda/D]$
$SU_{(N/M)} \downarrow SpO_{(N/M)}$	
$\{\lambda\}$	$\downarrow \langle\lambda/B\rangle$
$OSp_{(N/M)} \downarrow O_{(N)} \times Sp_{(M)}$	
$[\lambda]$	$\downarrow \sum_\zeta [\lambda/\zeta] \times \langle\tilde{\zeta}/B\rangle$
$SpO_{(N/M)} \downarrow Sp_{(N)} \times O_{(M)}$	
$\langle\lambda\rangle$	$\downarrow \sum_\zeta \langle\lambda/\zeta\rangle \times \langle\tilde{\zeta}/D\rangle$

Table 3. Dimensions of irreps of $OSp_{(N/M)}$.

$[\lambda]$	$D_{[\lambda]}$
$[1]$	x
$[2]$	$(1/2!)(x^2+y-2)$
$[1^2]$	$(1/2!)(x^2-y)$
$[3]$	$(x/3!)(x^2+3y-4)$
$[21]$	$(2x/3!)(x^2-4)$
$[1^3]$	$(x/3!)(x^2-3y+2)$
$[4]$	$(1/4!)(x^4+6x^2y-4x^2+3y^2-6y)$
$[31]$	$(3/4!)(x^4+2x^2y-y^2-2y-8x^2+8)$
$[2^2]$	$(2/4!)(x^4-10x^2+3y^2-6y)$
$[21^2]$	$(3/4!)(x^4-2x^2y-y^2-4x^2-2y)$
$[1^4]$	$(1/4!)(x^4-6x^2y+8x^2+3y^2-6y)$

Table 4. Dimensions of irreps of $SpO_{(N/M)}$.

$\langle\lambda\rangle$	$D_{\langle\lambda\rangle}$
$\langle 1\rangle$	x
$\langle 2\rangle$	$(1/2!)(x^2+y)$
$\langle 1^2\rangle$	$(1/2!)(x^2-y-2)$
$\langle 3\rangle$	$(x/3!)(x^2+3y+2)$
$\langle 21\rangle$	$(2x/3!)(x^2-1)$
$\langle 1^3\rangle$	$(x/3!)(x^2+3y-4)$
$\langle 4\rangle$	$(1/4!)(x^4+6x^2y+8x^2+3y^2+6y)$
$\langle 31\rangle$	$(3/4!)(x^4+2x^2y-y^2-6y-4x^2)$
$\langle 2^2\rangle$	$(2/4!)(x^4-10x^2+3y^2+6y)$
$\langle 21^2\rangle$	$(3/4!)(x^4-2x^2y-8x^2-y^2+2y+8)$
$\langle 1^4\rangle$	$(1/4!)(x^4-6x^2y-4x^2+3y^2-18y)$

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